Computer Science 294 Lecture 18 Notes

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1 Hypercontractivity II: Applications

1.1 Recap: Bonami's lemma and forms of hypercontractivity

Last time, we showed Bonami's lemma.

Lemma 1.1 (Bonami)**.** *Let $f : \{\pm 1\}^n \to \mathbb{R}$ have $\deg f \le k$, and let $X \sim \{\pm 1\}^n$. Then*

$$\mathbb{E}[f(X)^4] \le 9^k (\mathbb{E}[f(X)^2])^2.$$

We also saw two versions of hypercontractivity:

Theorem 1.1 ((4, 2)-hypercontractivity)**.** *For all $f : \{\pm 1\}^n \to \mathbb{R}$,*

$$\|T_{1/\sqrt{3}} f\|_4 \le \|f\|_2.$$

Theorem 1.2 ((2, 4/3)-hypercontractivity)**.** *For all $f : \{\pm 1\}^n \to \mathbb{R}$,*

$$\|T_{1/\sqrt{3}} f\|_2 \le \|f\|_{4/3}.$$

Here is a key corollary. Let $f : \{\pm 1\}^n \to \{\pm 1\}$, and let $g_i(x) = D_i f(x) \in \{-1, 0, 1\}$. Let the 1/3-stability be

$$\mathrm{Inf}_i^{(1/3)}(f) = \mathrm{Stab}_{1/3}(g) = \sum_{i \in S} \widehat{f}(S)^2 \left(\frac{1}{3}\right)^{|S|}.$$

Recall that

$$\mathrm{Inf}_i(f) = \sum_{i \in S} \widehat{f}(S)^2.$$

Last time, we showed the following.

Corollary 1.1.

$$\mathrm{Inf}_i^{(1/3)} \le \mathrm{Inf}_i(f)^{3/2}$$

Informally, if $\mathrm{Inf}_i(f) \ll 1$, then most "contribution" to $\sum_{i \in S} \widehat{f}(S)^2$ is from $\widehat{f}(S)$ with large $|S|$.

1.2 Friedgut's theorem

Now we will prove Friedgut's theorem. Here is the main lemma.

Lemma 1.2. *Let $f : \{\pm 1\}^n \to \{\pm 1\}$ and $\varepsilon > 0$. Choose $d = 2\mathbb{I}(f)/\varepsilon$, and let $\mathcal{J} = \{i \in [n] : \mathrm{Inf}_i(f) \geq 100^{-d}\}$. Then f is ε-concentrated on $\mathcal{F} := \{S \subseteq \mathcal{J} : |S| \leq d\}$.*

Proof. Our goal is to show that

$$\sum_{|S|>d} \widehat{f}(S)^2 + \sum_{|S|:S\not\subseteq\mathcal{J},|S|\leq d} \widehat{f}(S)^2 \leq \varepsilon.$$

We have

$$\begin{aligned}\sum_{|S|:|S|>d} &\leq \frac{1}{d}\sum_S \widehat{f}(S)^2|S| \\ &\leq \frac{1}{d}\mathbb{I}(f) \\ &\leq \varepsilon/2.\end{aligned}$$

It now remains to bound the second summation.

$$\sum_{|S|:S\not\subseteq\mathcal{J},|S|\leq d} \widehat{f}(S)^2 \leq \sum_{S\not\subseteq\mathcal{J},|S|\leq d} \widehat{f}(S)^2|S\cap\overline{\mathcal{J}}|$$

We want to use our key corollary from last lecture.

$$\begin{aligned}&= \sum_{|S|\leq d} \widehat{f}(S)^2 3^{|S|}\cdot\left(\frac{1}{3}\right)^{|S|}|S\cap\overline{J}| \\ &\leq 3^d \sum_{S\not\subseteq\mathcal{J},|S|\leq d} \widehat{f}(S)^2\left(\frac{1}{3}\right)^{|S|}|S\cap\overline{J}| \\ &\leq 3^d \sum_{i\in\overline{\mathcal{J}}}\sum_{S\ni i} \widehat{f}(S)^2\left(\frac{1}{3}\right)^{|S|} \\ &\leq 3^d \sum_{i\in\overline{\mathcal{J}}} \mathrm{Inf}_i^{(1/3)}(f)\end{aligned}$$

Using our key corollary,

$$\begin{aligned}&\leq 3^d \sum_{i\in\overline{\mathcal{J}}} \underbrace{\mathrm{Inf}_i(f)^{3/2}}_{=\mathrm{Inf}_i(f)\sqrt{\mathrm{Inf}_i(f)^{3/2}}} \\ &\leq \left(\frac{3}{10}\right)^d \sum_{i\in\overline{\mathcal{J}}} \mathrm{Inf}_i(f)\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{3}{10}\right)^d \mathbb{I}(f) \\
&\leq \varepsilon/2. \qquad \square
\end{aligned}
$$

Theorem 1.3 (Friedgut)**.** *Any function $f : \{\pm 1\}^n \to \{\pm 1\}$ is ε-close to a $2^{O(\mathbb{I}(f)/\varepsilon)}$-junta.*

Proof. Take $\mathcal{J}$ from the main lemma, and let

$$
g(s)L = \left(\sum_{S \subseteq \mathcal{J}} \widehat{f}(S)\chi_S(x)\right).
$$

We know that

$$
|\mathcal{J}| \leq \frac{\mathbb{I}(f)}{100^{-d}} \leq \mathbb{I}(f)2^{O(\mathbb{I}(f)/\varepsilon)} \leq 2^{O(\mathbb{I}(f)/\varepsilon)}.
$$

Then f is also ε-concentrated on $\{S \subseteq \mathcal{J}\}$: To check that g is a good approximation to f, we have

$$
\begin{aligned}
\mathbb{P}_{X \sim U_n}(f(X) \neq g(X)) &\leq \mathbb{E}_X \left[\left|f(X) - \sum_{S \subseteq \mathcal{J}} \widehat{f}(S)\chi_S(x)\right|\right] \\
&= \sum_{S : S \not\subseteq \mathcal{J}} \widehat{f}(S)^2 \\
&\leq \varepsilon. \qquad \square
\end{aligned}
$$

Corollary 1.2. *A width w CNF or DNF is ε-close to a $2^{O(w/\varepsilon)}$-junta.*

Remark 1.1. The junta in Friedgut's theorem can be a *restriction* of f; if $f : \{\pm 1\}^n \to \{pm1\}$ and we let $\mathscr{J}$ be such that $\sum_{S \subseteq \mathscr{J}} \widehat{f}(S)^2 \geq 1 - \varepsilon$, then there exists a restriction $(\mathscr{J}, z)$ is ε-close to f.

Proof. Choose $Z \sim \{\pm 1\}^{\overline{\mathcal{J}}}$. For each S,

$$
\mathbb{E}_Z[\widehat{f|_{J,Z}}(S)] = \begin{cases} 0 & S \not\subseteq \mathscr{J} \\ \widehat{f}(S) & \text{otherwise.} \end{cases}
$$

Then

$$
\begin{aligned}
\mathbb{E}_{Z \sim \{\pm 1\}^{\mathcal{J}}}[\langle f, f|_{\mathcal{J},Z}\rangle] &= \mathbb{E}_{Z \sim \{\pm 1\}^{\mathcal{J}}}[\langle \widehat{f}, \widehat{f|_{\mathcal{J},Z}}\rangle] \\
&= \sum_{S \subseteq [n]} \widehat{f}(S)\, \mathbb{E}_Z[\widehat{f|_{\mathcal{J},Z}}(S)] \\
&= \sum_{S \subseteq \mathcal{J}} \widehat{f}(S)^2
\end{aligned}
$$

$$\geq 1 - \varepsilon.$$

In particular, there must exist some z such that

$$1 - \varepsilon \leq \langle f, f|_{J,z} \rangle = 1 - 2\mathbb{P}_{X \sim U_n}(f(X) \neq f|_{J,z}(X)).$$

Rearranging, we get

$$\mathbb{P}_{X \sim U_n}(f(X) \neq f|_{J,z}(X)) \leq \varepsilon/2. \qquad \square$$

1.3 The KKL theorem

Theorem 1.4 (Kahn-Kalai-Linial)**.** *Let $f : \{\pm 1\}^n \to \{\pm 1\}$. Then*

$$\textit{Max Inf}(f) \geq \Omega(\mathrm{Var}(f) \tfrac{\log n}{n}).$$

Proof. Let $\varepsilon = \mathrm{Var}(f)/10$. We consider two cases:

1. If $\mathbb{I}(f) \geq \mathrm{Var}(f) \frac{\log n}{1000}$, then Max Inf$(f) \geq \mathrm{Var}(f) \frac{\log n}{1000n}$.
2. If $\mathbb{I}(f) < \mathrm{Var}(f) \frac{\log n}{1000}$, then $\mathbb{I}(f)/\varepsilon \leq \frac{\log n}{100}$. Using our main lemma, there exists a set $\mathcal{J}$ such that

   $$|\mathcal{J}| \leq 100^{2\mathbb{I}(f)/\varepsilon} \mathbb{I}(f) \leq 100^{2 \log n/100} \mathbb{I}(f) \leq 100^{\log n/50} \log n \leq \sqrt{n}$$

   Then

   $$\begin{aligned} \sum_{j \in \mathcal{J}} \mathrm{Inf}_j(f) &= \sum_{j \in \mathcal{J}} \sum_{S \ni j} \widehat{f}(S)^2 \\ &\geq \sum_{S \subseteq \mathscr{J}, |S| \geq 1} \widehat{f}(S)^2 \\ &\geq \sum_{|S| \geq 1} \widehat{f}(S)^2 - \sum_{S \subseteq \mathcal{J}} \widehat{f}(S)^2 \\ &\geq \mathrm{Var}(f) - \varepsilon \\ &\geq \frac{9}{10} \mathrm{Var}(f). \end{aligned}$$

   So there exists some $j \in \mathcal{J}$ such that

   $$\mathrm{Inf}_j(f) \geq \Omega\left(\frac{\mathrm{Var}(f)}{|\mathcal{J}|}\right) \geq \Omega\left(\frac{\mathrm{Var}(f)}{\sqrt{n}}\right) \geq \Omega\left(\frac{\mathrm{Var}(f) \log n}{n}\right). \qquad \square$$

Here is an application in social choice theory.

Corollary 1.3. *Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be monotone, and suppose that $\mathbb{E}_{X \sim U_n}[f(X)] \geq -0.99$. Then there exists a subset $\mathcal{J}$ of size at most $O(n/\log n)$ such that if all $j \in \mathcal{J}$ are "bribed to vote $+1$", then the outcome becomes $+1$ with high probability:*

$$\mathbb{E}_{X \sim \{\pm 1\}^{\overline{\mathcal{J}}}}[f(X) \mid \forall j \in \mathcal{J}, X_j = +1] \geq 0.99.$$

Proof sketch. We construct $\mathcal{J}$ by a greedy strategy: Let $f_0 := f$, so $\mathbb{E}[f_0] \geq -0.99$. If $\mathbb{E}[f_0] \geq 0.99$, we are done; otherwise, we know that $\mathbb{E}[f_0] \in (-0.99, +0.99)$. Then $\mathrm{Var}(f_0) = 1 - (\mathbb{E}[f_0])^2 \geq \Omega(1)$. By KKL, there exists an index $j_1 \in [n]$ such that $\mathrm{Inf}_{j_1}(f) \geq \Omega \log n/n$. Now include j_1 in our set $\mathcal{J}$, and let $f_1 := f_0^{j \mapsto +1}$. Then

$$\mathbb{E}[f_1] \geq \mathbb{E}[f_0] + \mathrm{Inf}_{j_1}(f_0) \geq \mathbb{E}[f_0] + \Omega(\tfrac{\log n}{n}).$$

Now if $\mathbb{E}[f_1] \geq 0.99$, we stop. Otherwise, we repeat.

This process stops at f_t, where

$$1 \geq \mathbb{E}[f_t] \geq \mathbb{E}[f_0] + \Omega(\tfrac{t \log n}{n}),$$

so we only need to do this process for $O(n/\log n)$ iterations. □

1.4 The FKN theorem

Theorem 1.5 (Friedgut-Kalai-Naor)**.** *If $f : \{\pm 1\}^n \to \{\pm 1\}$ has $W^{=1}(f) \geq 1 - \varepsilon$, then f is $O(\varepsilon)$-close to a dictator/antidictator.*

Remark 1.2. Another version of this theorem says that if $W^{\leq 1}(f) \geq 1 - \varepsilon$, then f is $O(\varepsilon)$-close to a 1-junta.

Proof. We linearize the function f: Let $\ell(x) = f^{=1}(x)$. We may assume without loss of generality that $W^{=1} = 1 - \varepsilon$. Then

$$\mathbb{E}[\ell(X)^2] = \sum_{i=1}^{n} \widehat{f}(\{i\})^2 = 1 - \varepsilon.$$

Then

$$\begin{aligned}
\ell(x)^2 &= \left(\sum_i \widehat{f}(\{i\}) x_i \right)^2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \widehat{f}(\{i\}) \widehat{f}(\{j\}) x_i x_j \\
&= \sum_{i=1}^{n} \widehat{f}(\{i\})^2 + 2 \sum_{1 \leq i < j \leq n} \widehat{f}(i) \widehat{f}(j) x_i x_j.
\end{aligned}$$

We want to bound the variance of $\ell(x)^2$, as it suffices to show that $\mathrm{Var}(\ell(X)^2) \leq O(\varepsilon)$. This is because we would get the lower bound $\sum_{i=1}^n \widehat{f}(\{i\})^4 \geq 1 - O(\varepsilon)$ because

$$1 - O(\varepsilon) \leq \left(\sum_{i=1}^n \widehat{f}(i)^2\right)^2 = \sum_{i=1}^n \widehat{f}(\{i\})^4 + 2\sum_{i\neq j} \widehat{f}(\{i\})^2 \widehat{f}(\{j\})^2.$$

This implies that

$$\max_i \widehat{f}(\{i\})^2 \cdot \sum_j \widehat{f}(\{j\})^2 \geq \sum_i \widehat{f}(\{i\})^4 \geq 1 - O(\varepsilon).$$

Now we bound the variance. We will use our anticoncentration bound:

$$\mathbb{P}_{X\sim U_n}(|\ell(X)^2 - (1-\varepsilon)| \geq \tfrac{1}{2}\sqrt{\mathrm{Var}(\ell^2)}) \geq \frac{(1-(\frac{1}{2})^2)^2}{9^2} \geq \frac{1}{144}.$$

If $\mathrm{Var}(\ell(X)^2) > 6400\varepsilon$, then

$$\mathbb{P}(|\ell(X)^2 - 1| \geq 39\sqrt{\varepsilon}) \geq \frac{1}{144}.$$

As a consequence, since $|\ell(X)^2 - 1| \geq 39\sqrt{\varepsilon}$ implies $(\ell(X)^2 f(X))^2 \geq 39\sqrt{\varepsilon}$,

$$\mathbb{P}((\ell(X)^2 f(X))^2 \geq 39\sqrt{\varepsilon}) \geq \frac{1}{144} \cdot 169\varepsilon > \varepsilon. \qquad \square$$